Section A: Pure Mathematics

1 The substitution $u = \cosh x$ should suggest itself (because of the factor of $\dfrac{du}{dx} = \sinh x$ in t numerator), and the resulting integral can be tackled by splitting the integrand into part fractions:

$$\int_0^a \frac{\sinh x}{2\cosh^2 x - 1}\,dx = \int_1^{\cosh a} \frac{du}{2u^2 - 1} = \frac{1}{2}\int_1^{\cosh a} \frac{1}{\sqrt{2}u - 1} - \frac{1}{\sqrt{2}u + 1}\,du$$

$$= \frac{1}{2\sqrt{2}}\left[\ln\left(\sqrt{2}u - 1\right) - \ln\left(\sqrt{2}u + 1\right)\right]_1^{\cosh a} = \frac{1}{2\sqrt{2}}\left(\ln\left(\frac{\sqrt{2}\cosh a - 1}{\sqrt{2}\cosh a + 1}\right) + \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)\right)$$

Similarly, substituting $u = \sinh x$, and then recognising an arctan integral:

$$\int_0^a \frac{\cosh x}{1 + 2\sinh^2 x}\,dx = \int_0^{\sinh a} \frac{du}{1 + 2u^2} = \frac{1}{\sqrt{2}}\left[\arctan\left(\sqrt{2}u\right)\right]_0^{\sinh a} = \frac{1}{\sqrt{2}}\arctan\left(\sqrt{2}\sinh a\right)$$

To show that

$$\int_0^\infty \frac{\cosh x - \sinh x}{1 + 2\sinh^2 x}\,dx = \frac{\pi}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right).$$

note that

a $\cosh^2 x = \sinh^2 x + 1$, so $2\cosh^2 x - 1 = 1 + 2\sinh^2 x$, and the integral required is t second minus the first of those calculated earlier, as $a \to \infty$.

b as $a \to \infty$, $\sinh a \to \infty$, so $\arctan\left(\sqrt{2}\sinh a\right) \to \dfrac{\pi}{2}$

c as $a \to \infty$, $\cosh a \to \infty$, so $\dfrac{\sqrt{2}\cosh a - 1}{\sqrt{2}\cosh a + 1} \to 1$ and $\ln\left(\dfrac{\sqrt{2}\cosh a - 1}{\sqrt{2}\cosh a + 1}\right) \to 0.$

Substituting $u = e^x$, so that $\cosh x = \frac{1}{2}\left(u + \dfrac{1}{u}\right)$ and $\sinh x = \frac{1}{2}\left(u - \dfrac{1}{u}\right)$:

$$\int_0^\infty \frac{\cosh x - \sinh x}{1 + 2\sinh^2 x}\,dx = \int_1^\infty \frac{\left(u + \frac{1}{u}\right) - \left(u - \frac{1}{u}\right)}{1 + \frac{1}{2}\left(u^2 - 2 + \frac{1}{u^2}\right)}\,\frac{1}{u}\,du = \int_1^\infty \frac{2}{1 + u^4}\,du$$

so $$\int_1^\infty \frac{1}{1 + u^4}\,du = \frac{\pi}{4\sqrt{2}} - \frac{1}{4\sqrt{2}}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right).$$

2 (i) Inspection of the denominator shows that the vertical asymptotes are at $x = 0$, $x = 4$, and the third term in $\mathrm{f}(x)$ tends to zero as $|x| \to \infty$, so the oblique asymptote is just $y = x - 4$.

The oblique asymptote meets the curve when $\dfrac{16\,(2x+1)^2}{x^2\,(x-4)} = 0$ or $(2x+1)^2 = 0$, hence there is a double root at $x = -\frac{1}{2}$ and hence the asymptote touches rather than crosses the curve at $(-\frac{1}{2}, -\frac{9}{2})$, so is a tangent there.

(ii) $\mathrm{f}(x) = 0$ when $x^2(x-4)^2 - 16(2x+1)^2 = 0$.

The left hand side of this equation is a difference of two squares, so factorises to give

$(x(x-4) - 4(2x+1))\,(x(x-4) + 4(2x+1)) = 0$; that is, $(x^2 - 12x - 4)\,(x+2)^2 = 0$, which has a double root at $x = -2$.

(iii) On your sketch you should show:

the double root at $(-2, 0)$ — the curve has a local maximum here and touches the x-axis;

the remaining roots (solutions of $x^2 - 12x - 4 = 0$) at $x = 6 \pm 2\sqrt{10}$;

the curve approaching the oblique asymptote $y = x - 4$ from below as $x \to \infty$, approaching it from above as $x \to -\infty$ and touching it at $(-\frac{1}{2}, -\frac{9}{2})$;

$\mathrm{f}(x) \to \infty$ as $x \to 0$ from above or below, $\mathrm{f}(x) \to +\infty$ as $x \to 4$ from below and $\mathrm{f}(x) \to -\infty$ as $x \to 4$ from above;

local minima at some x value with $0 < x < 4$ and with $y > 0$ and at some x value with $-2 < x < -\frac{1}{2}$ and with $-\frac{9}{2} > y > x - 4$ — note that this second minimum is not at the point of tangency with the oblique asymptote.

3 The sketch should show a curve with increasing gradient: because the gradient is increasing, the curve lies below the chord joining $(a, \mathrm{f}(a))$ and $(b, \mathrm{f}(b))$ and above the tangent to the curve at $\left(\frac{a+b}{2}, \mathrm{f}\left(\frac{a+b}{2}\right)\right)$. The illustration is clearer if $\mathrm{f}(x) > 0$ for $a \leqslant x \leqslant b$: then the area of the trapezium cut off by the chord and the lines $x = a$, $x = b$ and $y = 0$, which is $(b-a)\frac{\mathrm{f}(a)+\mathrm{f}(b)}{2}$, is larger than the area represented by the integral and the area of the trapezium cut off by the tangent and the lines $x = a$, $x = b$ and $y = 0$, which is $(b-a)\mathrm{f}\left(\frac{a+b}{2}\right)$, is smaller than the area represented by the integral.

Choose $\mathrm{f}(x) = \frac{1}{x^2}$, checking that this has $\mathrm{f}''(x) > 0$, $a = n-1$ and $b = n$ to get the quoted result.

Take the sum from $n = 2$ to ∞ of each term in the inequality: the left hand sum is directly as quoted; in the middle sum, you need to notice that it telescopes, so that all the terms except the first cancel in pairs; in the right hand sum, each reciprocal square occurs twice, cancelling the factor of $\frac{1}{2}$, except the first.

For the next part, observe that $\frac{1}{(n+1)^2} < \frac{1}{n^2}$, so $\frac{1}{2}\left(\frac{1}{n^2} + \frac{1}{(n+1)^2}\right) < \frac{1}{2}\left(\frac{1}{n^2} + \frac{1}{n^2}\right) = \frac{1}{n^2}$.

Finally, combine the two previous results to get

$$2\left(\frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots\right) < 1 < \frac{1}{2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots,$$

so that if $S = \sum_{n=1}^{\infty} \frac{1}{n^2}$, then $2\left(S - 1 - \frac{1}{2^2}\right) < 1 < S - \frac{1}{2}$; rearranging these inequalities gives the required bounds on S.

4 If circle n has centre O_n then $OO_n = \dfrac{r_n}{\sin\alpha}$, $OO_{n+1} = \dfrac{r_{n+1}}{\sin\alpha}$ and $OO_n - OO_{n+1} = r_n + r_{n+1}$.

Substituting and multiplying by $\sin\alpha$ gives $r_n - r_{n+1} = \sin\alpha(r_n + r_{n+1})$ which simplifies to the required result.

This result then implies that $r_n = \left(\dfrac{1-\sin\alpha}{1+\sin\alpha}\right)^n r_0$, so the total area is

$$S = \frac{1}{2}\pi r_0^2 + \pi\left(\left(\frac{1-\sin\alpha}{1+\sin\alpha}\right) r_0\right)^2 + \pi\left(\left(\frac{1-\sin\alpha}{1+\sin\alpha}\right)^2 r_0\right)^2 + \pi\left(\left(\frac{1-\sin\alpha}{1+\sin\alpha}\right)^3 r_0\right)^2 + \dots$$

which is almost a geometric series with common ratio $\left(\dfrac{1-\sin\alpha}{1+\sin\alpha}\right)^2$, so

$$S = \frac{\pi r_0^2}{1-\left(\dfrac{1-\sin\alpha}{1+\sin\alpha}\right)^2} - \frac{1}{2}\pi r_0^2 = \left(\frac{(1+\sin\alpha)^2}{4\sin\alpha} - \frac{1}{2}\right)\pi r_0^2 = \frac{1+\sin^2\alpha}{4\sin\alpha}\pi r_0^2.$$

Area T of triangle $OAB = \frac{1}{2}AB \times OO_0 = \dfrac{r_0}{\cos\alpha}\dfrac{r_0}{\sin\alpha}$,

so $\dfrac{S}{T} = \dfrac{\pi}{4}\cos\alpha\left(1+\sin^2\alpha\right) = \dfrac{\pi}{4}\cos\alpha\left(2-\cos^2\alpha\right)$.

By differentiation, the maximum $\dfrac{S}{T}$ occurs where $2 - 3\cos^2\alpha = 0$ (not $\sin\alpha = 0$) and equals

$$\frac{\pi}{4}\sqrt{\frac{2}{3}}\left(2-\frac{2}{3}\right) = \frac{\pi}{3}\sqrt{\frac{2}{3}} > \sqrt{\frac{2}{3}} = \sqrt{\frac{16}{24}} > \sqrt{\frac{16}{25}} = \frac{4}{5}.$$

5 If $\cos(x-\alpha) = \cos\beta$ then $x - \alpha = 2n\pi \pm \beta$ so $x = \alpha \pm \beta + 2n\pi$ so $\tan x = \tan(\alpha \pm \beta)$ however, for example, $x = \pi$, $\alpha = \beta = 0$ has $\tan x = \tan\pi = \tan 0 = \tan(\alpha+\beta)$ but $\cos(x-\alpha) = \cos\pi = -1 \neq 1 = \cos\beta$.

a Writing $\cos x - 7\sin x = R\cos(x-\alpha)$ requires $R = \sqrt{50} = 5\sqrt{2}$ and $\tan\alpha = -7$, so $\cos(x-\alpha) = \cos\beta$, where $\cos\beta = \dfrac{1}{\sqrt{2}}$, so we can take $\tan\beta = 1$.

Hence $\tan x = \tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \dfrac{-7 \pm 1}{1 \pm 7} = -\frac{3}{4}$ or $\frac{4}{3}$.

The first of these gives $x = \frac{1}{2}\pi + \omega$ or $x = \frac{3}{2}\pi + \omega$ (since $\arctan\frac{3}{4} = \frac{\pi}{2} - \arctan\frac{4}{3}$)and the second $x = \omega$ or $x = \pi + \omega$. However, the first solution in each case does not satisfy the original equation (both have $\sin x > 0$, so $\cos x - 7\sin x < 1$), so $x = \frac{3}{2}\pi + \omega$ or $\pi + \omega$.

b proceeding as in (i), $\cos(x-\alpha) = \cos\beta$, where $\tan\alpha = \frac{11}{2}$ and $R = 5\sqrt{5}$, so $\cos\beta = \dfrac{2}{\sqrt{5}}$ and so $\tan\beta = \frac{1}{2}$. Hence $\tan x = \dfrac{22 \pm 2}{4 \mp 11} = -\frac{24}{7}$ or $\frac{4}{3}$.

Notice that $\tan 2\omega = \dfrac{2\tan\omega}{1 - \tan^2\omega} = -\frac{24}{7}$ so the solutions are $x = \omega$ and $x = 2\omega$, again eliminating the other two possibilities, $\omega + \pi$ and $2\omega + \pi$, by checking in the original equation.

6 $F_n - F_{n-1} = w_n^2 + w_{n-1}^2 - 4w_n w_{n-1} - w_{n-1}^2 - w_{n-2}^2 + 4w_{n-1}w_{n-2} = w_n^2 - w_{n-2}^2 - 4w_{n-1}(w_n - w_{n-2})$
$= (w_n - w_{n-2})(w_n + w_{n-2} - 4w_{n-1}.)\quad (+)$

(i) Let w_n be u_n; then $u_n + u_{n-2} - 4u_{n-1} = 0$, so $F_n - F_{n-1} = 0$ for $n \geqslant 2$, by (+), but $F_1 = u_1^2 + u_0^2 - 4u_1u_0 = -3$ so $F_n = -3$ for $n \geqslant 1$

(ii) In this part, let w_n be v_n.

(a) $v_1^2 + 1 = 4v_1 - 3 \Rightarrow (v_1 - 2)^2 = 0 \Rightarrow v_1 = 2$
$F_n = v_n^2 + v_{n-1}^2 - 4v_nv_{n-1} = -3$ for $n \geqslant 1$
$\Rightarrow v_n - v_{n-2} = 0$ or $v_n + v_{n-2} - 4v_{n-1} = 0$, for $n \geqslant 2$, by (+).

(b) Since $1, 2, 1, 2, \ldots$ satisfies $v_n - v_{n-2} = 0$ for $n \geqslant 2$, F_n is constant, by (+) and since $v_0 = 1, v_1 = 2$ that constant is -3, so the sequence satisfies $(*)$.

(c) The sequence $1, 2, 7, 2, \ldots$, with period 4, satisfies $v_n - v_{n-2} = 0$ for odd $n \geqslant 2$ and $v_n + v_{n-2} - 4v_{n-1} = 0$ for even $n \geqslant 2$, so F_n is constant, by (+), and since $v_0 = 1, v_1 = 2$ that constant is -3, so the sequence satisfies $(*)$.

7 On $0 \leqslant t \leqslant 1$, the integrand is non-negative and $0 \leqslant \dfrac{t}{t+1} = 1 - \dfrac{1}{t+1} \leqslant \frac{1}{2}$, so

$$I_{n+1} = \int_0^1 \frac{t^n}{(t+1)^{n+1}}\,\mathrm{d}t = \int_0^1 \frac{t}{t+1}\frac{t^{n-1}}{(t+1)^n}\,\mathrm{d}t < \frac{1}{2}\int_0^1 \tfrac{1}{2}\frac{t^{n-1}}{(t+1)^n}\,\mathrm{d}t = \tfrac{1}{2}I_n.$$

Integration by parts gives

$$I_{n+1} = \left[-\frac{t^n}{n(t+1)^n}\right]_0^1 + \int_0^1 \frac{nt^{n-1}}{n(t+1)^n}\,\mathrm{d}t = -\frac{1}{n2^n} + I_n,$$

so $I_n > 2I_{n+1} = -\dfrac{1}{n2^{n-1}} + 2I_n \quad \Rightarrow \quad I_n < \dfrac{1}{n2^{n-1}} \quad (*)$.

Since $\dfrac{1}{2^r} = I_r - I_{r+1}$, $\displaystyle\sum_{r=1}^{n}\frac{1}{r2^r} = (I_1 - I_2) + (I_2 - I_3) + \ldots + (I_n - I_{n+1}) = I_1 - I_{n+1}$, and $I_1 = \displaystyle\int_0^1 \frac{1}{t+1}\,\mathrm{d}t = \ln 2$, so $\ln 2 = \displaystyle\sum_{r=1}^{n}\frac{1}{r2^r} + I_{n+1}$.

Hence $\ln 2 > \displaystyle\sum_{r=1}^{3}\frac{1}{r2^r} = \frac{2}{3}$ and, by inequality $(*)$, $\ln 2 = \displaystyle\sum_{r=1}^{2}\frac{1}{r2^r} + I_3 < \sum_{r=1}^{2}\frac{1}{r2^r} + \frac{1}{3\cdot 2^2} = \frac{17}{24}$.

8 If $u = y^2$ then $\dfrac{\mathrm{d}u}{\mathrm{d}x} = 2y\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2\mathrm{f}(x)y^2 + 2\mathrm{g}(x) = 2\mathrm{f}(x)u + 2\mathrm{g}(x)$,

which is a linear differential equation for $u(x)$.

In this case, $\mathrm{f}(x) = \dfrac{1}{x}$, $\mathrm{g}(x) = -1$ so the differential equation is $\dfrac{\mathrm{d}u}{\mathrm{d}x} = \dfrac{2u}{x} - 2$.

The integrating factor is $\mathrm{e}^{\int \frac{2}{x}\mathrm{d}x} = \mathrm{e}^{2\ln x} = \dfrac{1}{x^2}$, giving $\dfrac{1}{x^2}\dfrac{\mathrm{d}u}{\mathrm{d}x} - \dfrac{2u}{x^3} = \dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{u}{x^2}\right) = \dfrac{-2}{x^2}$

so that $\dfrac{u}{x^2} = \displaystyle\int \frac{-2}{x^2}\mathrm{d}x = \frac{2}{x} + c$ or $u = y^2 = cx^2 + 2x$.

The solution curves which pass through $(1,\ 1)$, $(2,\ 2)$ and $(4,\ 4)$ are $y^2 + (x-1)^2 = 1$, $y^2 = 2x$ and $(x+2)^2 - 2y^2 = 4$ respectively. In drawing these curves it should be made clear that all of them pass through the origin, and that this is their only point of intersection; that the first is a circle with centre $(1, 0)$, the second a parabola and the third an hyperbola with centre $(-2, 0)$ and asymptotes $y = \pm\dfrac{x+2}{\sqrt{2}}$.

Section B: Mechanics

9 Let angle AOM be 2θ, so APM $= \theta$, and let R, F be the normal reaction and frictional forces of the hoop on the mouse.

The forces on the hoop are its weight, the force on the hoop from its suspension, and the reaction on the hoop to the forces R and F of the hoop on the mouse. For the hoop to be in equilibrium, the net moment of these forces about the point of suspension must be zero, but the lines of action of the weight of the hoop, and the force on it from its suspension, pass through the point of suspension, so have zero moment about it. Thus equilibrium of hoop requires the net moment of the reactions to R and F about the point of suspension of the hoop to be zero; that is, $F \times \text{PM}\cos\theta - R \times \text{PM}\sin\theta = 0$, or $F = R\tan\theta$.

For the mouse (of mass m, say) to have constant speed u, its equations of motion are:

resolving radially inward, $R - mg\cos 2\theta = \dfrac{mu^2}{a}$ and resolving tangentially, $F - mg\sin 2\theta = 0$. Combining these three equations gives $mg\sin 2\theta\cos\theta = \left(mg\cos 2\theta + \dfrac{mu^2}{a}\right)\sin\theta$ which reduces to $u^2 = ag$ using the double angle identities.

To maintain a speed u with $u^2 = ag$ requires $R = mg(\cos 2\theta + 1) = 2mg\cos^2\theta$ and $F = mg\sin 2\theta = 2mg\cos^2\theta\tan\theta$ which is greater than μR if θ exceeds $\arctan\mu$ and hence angle AOM exceeds $2\arctan\mu$, so, initially, the hoop will begin to rotate in the opposite sense to the mouse's motion round the circle.

10 For $6a \leqslant x \leqslant 7a$, $\ddot{x} = g + 6g\dfrac{(7a-x)}{2a} - 6g\dfrac{(x-6a)}{6a} = \dfrac{4g}{a}(7a-x)$

and for $7a \leqslant x \leqslant 9a$, $\ddot{x} = g - 6g\dfrac{(x-6a)}{6a} = \dfrac{g}{a}(7a-x)$.

Notice that these both describe simple harmonic motion with $x = 7a$ as the equilibrium position so that, for $6a \leqslant x \leqslant 7a$,

$$x = 7a + A\cos\sqrt{\frac{4g}{a}}t + B\sin\sqrt{\frac{4g}{a}}t \text{ and } \dot{x} = -\sqrt{\frac{4g}{a}}A\sin\sqrt{\frac{4g}{a}}t + \sqrt{\frac{4g}{a}}B\cos\sqrt{\frac{4g}{a}}t$$

and initial conditions $x = 6a, \dot{x} = 0$ at $t = 0$ then give $A = -a, B = 0$.

Let the particle pass through $x = 7a$ at $t = t_0$; then $\sqrt{\dfrac{4g}{a}}t_0 = \dfrac{\pi}{2}$ and, at this point, $\dot{x} = \sqrt{4ga}$.

For $7a \leqslant x \leqslant 9a$, similarly $x = 7a + A\cos\sqrt{\dfrac{g}{a}}(t-t_0) + B\sin\sqrt{\dfrac{g}{a}}(t-t_0)$. The initial conditions are $x = 7a, \dot{x} = \sqrt{4ga}$ at $t = t_0$, which give $A = 0, B = 2a$.

Finally, $x = 9a$ when $\sqrt{\dfrac{g}{a}}(t-t_0) = \dfrac{\pi}{2}$; that is, when $t = \dfrac{\pi}{2}\sqrt{\dfrac{a}{4g}} + \dfrac{\pi}{2}\sqrt{\dfrac{a}{g}} = \dfrac{3\pi}{4}\sqrt{\dfrac{a}{g}}$.

11 Since z is initially, and hence always, positive, Newton's Law gives $2\ddot{x_1} = -\dfrac{2}{z^3}$ and $\ddot{x_2} = \dfrac{2}{z^3}$, so that $\ddot{z} = \ddot{x_2} - \ddot{x_1} = \dfrac{3}{z^3}$.

Writing this equation as $v\dfrac{\mathrm{d}v}{\mathrm{d}z} = \dfrac{3}{z^3}$ and integrating with respect to z, we have $\dfrac{v^2}{2} = -\dfrac{3}{2z^2} + c$ so $v = \pm\sqrt{2c - \dfrac{3}{z^2}}$ where the initial condition $v = -1$ when $z = 1$ requires the negative sign to be chosen and c=2.

Writing $v = \dfrac{\mathrm{d}z}{\mathrm{d}t}$ and separating the variables gives

$$\int \frac{\mathrm{d}z}{\sqrt{4 - \frac{3}{z^2}}} = -\int \mathrm{d}t \quad \text{or} \quad c - t = \int \frac{z\mathrm{d}z}{\sqrt{4z^2 - 3}} = \tfrac{1}{4}\sqrt{4z^2 - 3}$$

so that $\sqrt{4z^2 - 3} = 1 - 4t$, using the initial condition $z = 1$ at $t = 0$ to determine c.

Then $z = \sqrt{4t^2 - 2t + 1}$ as required.

Defining $w = x_2 + 2x_1$, $\ddot{w} = \ddot{x_2} + 2\ddot{x_1} = 0$ so that $\dot{w} = a$, $w = at + b$. Initially, $\dot{x_1} = 1$ and $\dot{x_2} = 0$ so $a = 2$; $x_1 = 0$ and $x_2 = 1$ so $b = 1$. This gives

$x_1 = \frac{1}{3}(w - z) = \frac{1}{3}\left(2t + 1 - \sqrt{4t^2 - 2t + 1}\right)$ and $x_2 = \frac{1}{3}(w + 2z) = \frac{1}{3}\left(2t + 1 + 2\sqrt{4t^2 - 2t + 1}\right)$.

It is worth noting, though not required by the question, that $\dot{x_1} \to \frac{1}{2}$, $\dot{x_2} \to 2$ as $t \to \infty$.

Section C: Statistics

12 For C_1, we have $\mathrm{P}(0) = \dfrac{m-1}{m}$ and $\mathrm{P}(1) = \dfrac{1}{m}$, so that $\mathrm{E}\,[C_1] = 0 \times \dfrac{m-1}{m} + 1 \times \dfrac{1}{m} = \dfrac{1}{m}$ and $\mathrm{Var}\,[C_1] = \left(0^2 \times \dfrac{m-1}{m} + 1^2 \times \dfrac{1}{m}\right) - \left(\dfrac{1}{m}\right)^2 = \dfrac{m-1}{m^2}$

$\mathrm{Cov}\,[C_1, C_2] = 1^2 \times \mathrm{P}(C_1 = C_2 = 1) - \mathrm{E}[C_1]\mathrm{E}[C_1]$ (since the other terms in the expectation of C_1C_2 are all zero). $\mathrm{P}(C_1 = C_2 = 1) = \mathrm{P}(\text{players 1 and 2 get their own shirts}) = \dfrac{1}{m}\dfrac{1}{m-1}$,

so $\mathrm{Cov}\,[C_1, C_2] = \dfrac{1}{m(m-1)} - \left(\dfrac{1}{m}\right)^2 = \dfrac{1}{m^2(m-1)}$

$\mathrm{E}\,[N] = \mathrm{E}\,[C_1] + \mathrm{E}\,[C_2] + \ldots = m \cdot \dfrac{1}{m} = 1$ and $\mathrm{Var}\,[N] = \mathrm{Var}\,[C_1] + \mathrm{Var}\,[C_2] + \ldots + \mathrm{Cov}\,[C_1, C_2] + \mathrm{Cov}\,[C_1, C_3] + \mathrm{Cov}\,[C_2, C_1] + \ldots = m \cdot \mathrm{Var}\,[C_1] + m(m-1) \cdot \mathrm{Cov}\,[C_1, C_2] = 1$.

A normal approximation with mean and standard deviation both equal to 1 is not likely to be appropriate as the approximation would give high probability to negative values of N, which are impossible. A Poisson approximation might be reasonable as mean = variance.

There are 9 arrangements where no player wears his own shirt out of 24 permutations,while the Poisson approximation to $\mathrm{P}(0)$, with mean 1, is e^{-1}.

The relative error is $\dfrac{\frac{9}{24} - \mathrm{e}^{-1}}{\frac{9}{24}} \approx 1 - \dfrac{800}{3 \times 272} = \dfrac{2}{102} \approx 2\%$.

13 (i) P(a competitor drops out in round r) $= p^{r-1}(1-p)$

so P(all three drop out in round r)$= \left(p^{r-1}(1-p)\right)^3$,

so P(all three drop out in the same round) $\equiv P_3 = \sum_{r=1}^{\infty}\left(p^{r-1}(1-p)\right)^3$

This is a geometric series with common ratio p^3 and first term $(1-p)^3$

so $P_3 = \dfrac{(1-p)^3}{1-p^3}$.

(ii) The probability that a competitor survives round $r-1$ is p^{r-1}, so the probability that a competitor drops out in round $r-1$ or earlier (that is, before round r) is $1-p^{r-1}$. Therefore the probability that two competitors drop out in round r and the third earlier is $3 \times \left(p^{r-1}(1-p)\right)^2 \times \left(1-p^{r-1}\right)$, where the factor of three is required, because any of the three could be the one to drop out earliest.

(iii) From (ii), Pr(two drop out in same round and the third earlier) $\equiv P_2$

$= \sum_{r=2}^{\infty} 3\left(p^{r-1}(1-p)\right)^2\left(1-p^{r-1}\right) = 3(1-p)^2\sum_{r=2}^{\infty}\left(p^{2(r-1)} - p^{3(r-1)}\right)$

$= 3(1-p)^2\left(\dfrac{p^2}{1-p^2} - \dfrac{p^3}{1-p^3}\right)$, summing to infinity two geometric series with first terms p^2 and p^3 and common ratios p^2 and p^3 respectively.

Pr(the grand prize is awarded) $= 1 - P_2 - P_3$, which simplifies to $\dfrac{3p\left(1+p^2\right)}{(1+p)\left(1+p+p^2\right)}$, using the factorisation $1-p^3 = (1-p)(1+p+p^2)$.

14 The test is appropriate because, if H_0 were true, $\bar{x}$ would have a higher probability of being in the region stated than if H_1 were true.

Under H_0, $\bar{X}$ has a Normal distribution with mean μ and standard deviation $\dfrac{\sigma_0}{\sqrt{n}}$ so

$$\alpha = P(|\bar{X} - \mu| > c) = 2\left(1 - \Phi\left(\frac{c}{\frac{\sigma_0}{\sqrt{n}}}\right)\right)$$

$$\text{so } \Phi\left(\frac{c}{\frac{\sigma_0}{\sqrt{n}}}\right) = 1 - \frac{\alpha}{2}, \text{ so } \frac{c}{\frac{\sigma_0}{\sqrt{n}}} = z_\alpha \text{ or } c = \frac{\sigma_0 z_\alpha}{\sqrt{n}}.$$

Under H_1, $\bar{X}$ has a Normal distribution with mean μ and standard deviation $\dfrac{\sigma_1}{\sqrt{n}}$ so

$$\beta = P(|\bar{X} - \mu| < c) = 1 - 2\left(1 - \Phi\left(\frac{c}{\frac{\sigma_1}{\sqrt{n}}}\right)\right) = 2\Phi\left(\frac{c}{\frac{\sigma_1}{\sqrt{n}}}\right) - 1 = 2\Phi\left(\frac{\sigma_0 z_\alpha}{\sigma_1}\right) - 1,$$

so β is independent of n.

$$\beta < 0.05 \quad \Rightarrow \quad \Phi\left(\frac{\sigma_0 z_\alpha}{\sigma_1}\right) < \frac{1 + 0.05}{2} = 0.525 \quad \Rightarrow \quad \frac{\sigma_0 z_\alpha}{\sigma_1} < 0.063.$$

$$\alpha < 0.05 \quad \Rightarrow \quad z_\alpha > 1.960.$$

For these both to hold, we must have $0.063 > \dfrac{\sigma_0 z_\alpha}{\sigma_1} > \dfrac{1.960\sigma_0}{\sigma_1}$ or $\dfrac{\sigma_1}{\sigma_0} > \dfrac{1.960}{.063} = \dfrac{280}{9} \approx 30.$